Baryon Mass Splittings in the $1/N_c$ Expansion

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The $I = 0, 1, 2, 3$ mass splittings of the spin-1/2 octet and spin-3/2 decuplet baryons are analyzed in the $1/N_c$ expansion combined with perturbative flavor breaking. We show there is considerable experimental evidence that the baryon masses satisfy the hierarchy predicted by this expansion. Since flavor symmetry-breaking suppression factors alone are not sufficient to describe the observed hierarchy, we conclude that there is firm evidence for the $1/N_c$ expansion in the baryon masses. Our analysis differs from non-relativistic $SU(6)$.

I. INTRODUCTION

The $1/N_c$ expansion has led to new understanding of the spin-flavor structure of baryons in QCD [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. In the large N_c limit [16,17], it has been shown that the baryon sector of QCD possesses a contracted spin-flavor algebra [1,18]. Corrections to the large N_c limit can be parametrized by $1/N_c$-suppressed operators with definite spin-flavor transformation properties [1,2]. By studying the spin-flavor structure of these $1/N_c$ corrections, it is possible to obtain new symmetry relations which are satisfied to non-trivial orders in the $1/N_c$ expansion, where the accuracy of these relations is predicted by the expansion. In the cases studied thus far, the $1/N_c$ expansion has yielded predictions for static properties of baryons which agree with the experimental data at the level of precision predicted by the expansion.

In this paper we analyze the isospin mass splittings of the spin-1/2 octet and spin-3/2 decuplet baryons in the $1/N_c$ expansion, with isospin symmetry breaking treated perturbatively. $SU(3)$ flavor symmetry breaking is treated first perturbatively, and then nonperturbatively through the use of $SU(2)_I \times U(1)_Y$ flavor symmetry. We find that there is evidence for the pattern of mass splittings predicted by the $1/N_c$ expansion and flavor symmetry breaking. A number of our predictions are not tested by the experimental data because they involve baryon mass splittings which are poorly measured; more accurate measurements of these splittings would test the validity of the expansion. The analysis we perform in this work illustrates that the predictions of the $1/N_c$ expansion are different from the standard $SU(6)$ predictions of the non-relativistic quark model [19,20].

The analysis of the isospin mass splittings in the $1/N_c$ expansion is organized as follows. Section II presents the relevant operator analysis. $1/N_c$ expansions are constructed for the $SU(2) \times SU(3)$ representations $(0,1)$, $(0,8)$, $(0,27)$, $(0,64)$ and $(0, 10 + \overline{10})$. In Sec. III, we give the complete set of linearly independent operators which spans the baryon masses for $N_c = 3$. Each operator occurs at a particular order in $1/N_c$ and flavor symmetry breaking. Mass relations for the octet and decuplet baryons are derived in Secs. IV and V, where we separate the relations into isospin channels $I = 0$, 1, 2 and 3. In Sec. IV we present the analysis with perturbative $SU(3)$ flavor symmetry breaking; Sec. V repeats the analysis using $SU(2) \times U(1)$ flavor symmetry. We contrast the results of the two analyses, and comment on their relation to $SU(6)$ formulæ. Our conclusions are presented in Sec. VI.

II. OPERATOR ANALYSIS

The lowest-lying baryons for N_c colors transform according to the completely symmetric spin-flavor $SU(2F)$ representation shown in Fig. 1. This baryon representation decomposes under (spin ⊗ flavor) into a tower of baryon states with spins $J = 1/2, 3/2, \ldots, N_c/2$, with the respective flavor representations displayed in Fig. 2. In the following analysis, we consider the special case of $F = 3$ light flavors. For three light flavors, the flavor representations of Fig. 2 for N_c large and finite differ from the flavor representations for $N_c = 3$. The weight diagrams for the flavor representations of the spin-1/2 and spin-3/2 baryons are given in Figs. 3 and 4, respectively. These flavor representations reduce to the baryon octet and decuplet for $N_c = 3$. For arbitrary N_c, the familiar spin-1/2 and spin-3/2 baryons can be identified with states at the top of the flavor representations, for which the number of strange quarks is $O(1)$ (not $O(N_c)$). In the following analysis, we are only interested in the masses of baryon states which continue to exist for $N_c = 3$; we

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call these states the physical baryons. Focusing on these baryons results in a number of simplifications in the analysis. We caution the reader that the $1/N_c$ analysis we perform here for arbitrary N_c is valid only for the physical baryons in Figs. 3 and 4, not the entire flavor representations. The results are also valid if the number of colors is set equal to three.

We construct an operator expansion for the mass splittings of the baryon octet and decuplet using quark operators as the operator basis. Equivalent results can be derived in the Skyrme operator basis. The complete classification of quark operators for three flavors was performed in Ref. [12]; in this work we use the same notation. The sole zero-body operator is denoted by $\mathbb{1}$, and the complete set of irreducible one-body operators is denoted by

$$
\begin{aligned}
J^i &= q^\dagger \left(J^i \otimes \mathbb{1} \right) q & (1,1), \\
T^a &= q^\dagger \left(\mathbb{1} \otimes T^a \right) q & (0,8), \\
G^{ia} &= q^\dagger \left(J^i \otimes T^a \right) q & (1,8),
\end{aligned}
\qquad (2.1)
$$

where J^i are the spin generators, T^a are the flavor generators, and G^{ia} are the spin-flavor generators. These operators transform as irreducible representations of $SU(2) \times SU(3)$, which are denoted in Eq. (2.1) by the spin J and the dimension of the $SU(3)$ flavor representation.

The $1/N_c$ expansion of any operator transforming according to a given $SU(2) \times SU(3)$ representation is given by an expansion of the form

$$
\mathcal{O} = \sum_n c_n \frac{1}{N_c^{n-1}} \mathcal{O}_n, \qquad (2.2)
$$

where the n-body operators $\mathcal{O}_n$ are of the generic form

$$
\frac{1}{N_c^{n-1}} \mathcal{O}_n = N_c \left(\frac{J^i}{N_c} \right)^l \left(\frac{T^a}{N_c} \right)^m \left(\frac{G^{jb}}{N_c} \right)^{n-l-m}, \quad (2.3)
$$

and the c_n are unknown coefficients. The number of operators participating in the expansion (2.2) can be reduced to a finite set using operator identities. The operator classification of Ref. [12] showed that the complete set of linearly independent quark operators for the baryon representation Fig. 1 transforms according to the irreducible spin-flavor representations

$$
1 + T_\beta^\alpha + T_{(\beta_1 \beta_2)}^{(\alpha_1 \alpha_2)} + \ldots + T_{(\beta_1 \beta_2 \ldots \beta_{N_c})}^{(\alpha_1 \alpha_2 \ldots \alpha_{N_c})}, \qquad (2.4)
$$

where $T_{(\beta_1 \beta_2 \ldots \beta_m)}^{(\alpha_1 \alpha_2 \ldots \alpha_m)}$ is a traceless tensor completely symmetric in its m upper and lower indices. Ref. [12] proved that the spin-flavor representation $T_{(\beta_1 \beta_2 \ldots \beta_m)}^{(\alpha_1 \alpha_2 \ldots \alpha_m)}$ consists of purely m-body quark operators, since all n-body quark operators ($n > m$) transforming according to this tensor representation can be reduced to m-body operators using non-trivial operator identities. The expansion (2.4) terminates at N_c-body operators, since no higher than N_c-body operators are required to describe any spin-flavor operator acting on an N_c-quark baryon.

The above analysis implies that the complete $1/N_c$ expansion of any operator transforming according to a given $SU(2) \times SU(3)$ representation can be written in terms of pure n-body operators, which transform according to definite $SU(6)$ representations. This set of operators is not the natural basis which arises in an expansion in flavor symmetry breaking, however. Instead, the natural basis consists of operators which are associated with definite powers of flavor symmetry-breaking parameters; such operators have no contracted flavor indices. The operators in this new basis are not pure n-body operators, but contain components which can be reduced to lower-body operators using non-trivial operator identities. Thus, the operators which are natural with regard to the flavor-breaking analysis do not always correspond to definite $SU(6)$ representations, and the results we obtain differ from $SU(6)$ formulæin some instances.

In the following analysis, we are only interested in the predictions of the $1/N_c$ expansion for the baryons that exist for $N_c = 3$, namely the spin-1/2 octet and spin-3/2 decuplet baryons. When the set of baryon states considered for large N_c is restricted to these physical baryons, all n-body operators with $n > 3$ are redundant and linearly dependent on 0-, 1-, 2- and 3-body operators. Thus, the complete set of independent operators acting on this restricted set of baryon states for any N_c is given by the spin-flavor representations

$$
\begin{aligned}
&\left(\boxed{} \otimes \boxed{} \right) \\
&= 1 + T_\beta^\alpha + T_{(\beta_1 \beta_2)}^{(\alpha_1 \alpha_2)} + T_{(\beta_1 \beta_2 \beta_3)}^{(\alpha_1 \alpha_2 \alpha_3)} \\
&= 1 + 35 + 405 + 2695, \qquad (2.5)
\end{aligned}
$$

where the dimensions of the $SU(6)$ representations are given in the last line of Eq. (2.5).

To analyze the mass splittings of the octet and decuplet baryons, we need the spin-zero $SU(2) \times SU(3)$ representations of the quark operators contained in the $SU(6)$ representations 1, 35, 405, and 2695. As is well known, the 1 contains a $(0,1)$; the 35 contains a $(0,8)$; the 405 contains $(0,1)$, $(0,8)$ and $(0,27)$; and the 2695 contains $(0,8)$, $(0,27)$, $(0,64)$ and $(0,10+\overline{10})^*$. If we were interested in n-body operators for $n > 3$, there would be additional $SU(3)$ representations to consider. For example, the purely 4-body $SU(6)$ representation contains $(0,1)$, $(0,8)$, $(0,27)$, $(0,27)$, $(0,64)$, $(0,125)$ and $(0,35+\overline{35})$. All of these quark operators either vanish or reduce to 0-, 1-, 2- and 3-body operators *when one restricts the set of baryons of interest to the physical baryon states.*

Thus, in the analysis of the isospin splittings of the octet and decuplet baryons, we need to construct $1/N_c$ expansions only for the $SU(2) \times SU(3)$ representations

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*The representation $(0, 10 - \overline{10})$ is not allowed by time reversal invariance.

$(0, 1)$, $(0, 8)$, $(0, 27)$, $(0, 64)$, and $(0, 10 + \overline{10})$. Furthermore, the $1/N_c$ expansions can be truncated at three-body operators when one considers only physical baryon states. The set of 0-, 1-, 2- and 3-body operators in the $1/N_c$ expansions spans all of the mass splittings of the physical baryons. Only if one is able to truncate the $1/N_c$ expansions before the occurrence of three-body operators does one obtain any non-trivial mass relations which are valid to a given order in the $1/N_c$ expansion. $1/N_c$ operator expansions up to 3-body operators are obtained for the $(0, 1)$, $(0, 8)$, $(0, 27)$, $(0, 64)$, and $(0, 10 + \overline{10})$ representations in the following subsections.

A. $(0, 1)$

The complete set of $(0, 1)$ operators can be classified using the operator identities of Ref. [12]. There is only one zero-body operator transforming as $(0, 1)$ under $SU(2) \times SU(3)$,

$$\mathcal{O}_0 = \mathbb{1}, \tag{2.6}$$

and only one two-body operator,

$$\mathcal{O}_2 = J^2 . \tag{2.7}$$

In general, there is a single n-body operator for each even n,

$$\mathcal{O}_n = J^n , \tag{2.8}$$

which transforms as $(0, 1)$. Thus, the general $1/N_c$ expansion for a $(0, 1)$ operator is of the form [2,4,5,6,12].

$$\mathcal{O} = \sum_{n=0,2}^{N_c - 1} c_n \frac{1}{N_c^{n-1}} \mathcal{O}_n , \tag{2.9}$$

for N_c odd, where c_n are unknown coefficients. Since the matrix elements of J are order one for baryons with spins at the bottom of the tower [1], each succeeding operator is suppressed by relative order $1/N_c^2$, and so it is possible to truncate the $1/N_c$ expansion (2.9) at any desired point in the expansion.

B. $(0, 8)$

The $1/N_c$ expansion for a $(0, 8)$ operator appears in Ref. [12]. Operator reduction identities imply that only n-body operators with a single factor of either T^a or G^{ia} need to be retained. There is only one one-body operator,

$$\mathcal{O}_1^a = T^a, \tag{2.10}$$

and only one two-body operator,

$$\mathcal{O}_2^a = \{J^i, G^{ia}\} , \tag{2.11}$$

allowed after operator reduction. In general, there is only one independent n-body operator for each n. All of these operators can be generated recursively from $\mathcal{O}_1^a$ and $\mathcal{O}_2^a$ by anticommuting with J^2,

$$\mathcal{O}_{n+2}^a = \{J^2, \mathcal{O}_n^a\} . \tag{2.12}$$

The set of operators $\mathcal{O}_n^a$, $n = 1, 2, \ldots, N_c$, forms a complete set of linearly independent spin-zero octets. Thus, the $1/N_c$ expansion for a $(0, 8)$ has the form

$$\mathcal{O}^a = \sum_{n=1,2}^{N_c} c_n \frac{1}{N_c^{n-1}} \mathcal{O}_n^a, \tag{2.13}$$

where c_n are unknown coefficients. Since J is of order one, the operator $\mathcal{O}_{n+2}^a$ in Eq. (2.13) is suppressed by $1/N_c^2$ relative to $\mathcal{O}_n^a$. Thus, it is valid to truncate the expansion (2.13) for arbitrary a after the first two terms, up to corrections of relative order $1/N_c^2$. Since for $N_c = 3$, the expansion Eq. (2.13) ends with the three-body operator

$$\mathcal{O}_3^a = \{J^2, T^a\}, \tag{2.14}$$

this truncation amounts to the neglect of one operator.

There are two different $(0, 8)$ operators which are relevant for the analysis of the baryon mass splittings: $\mathcal{O}^8$ with $I = 0$ and $\mathcal{O}^3$ with $I = 1$. It is important to further analyze the implicit N_c-dependence of the matrix elements of $\mathcal{O}_1^a$ and $\mathcal{O}_2^a$ for the physical baryons in these special cases. The matrix elements of the one-body operators T^a and G^{ia}, $a = 8$ and $a = 3$, can be rewritten in terms of quark number and spin operators,

$$
\begin{aligned}
T^8 &= \frac{1}{2\sqrt{3}} \left(N_c - 3N_s\right), \\
G^{i8} &= \frac{1}{2\sqrt{3}} \left(J^i - 3J_s^i\right), \\
T^3 &= \frac{1}{2} \left(N_u - N_d\right), \\
G^{i3} &= \frac{1}{2} \left(J_u^i - J_d^i\right),
\end{aligned}
\tag{2.15}
$$

where $N_c = N_u + N_d + N_s$ and $J^i = J_u^i + J_d^i + J_s^i$. Using Eq. (2.15), one finds the matrix elements

$$
\begin{aligned}
\{J^i, G^{i8}\} &= \frac{1}{2\sqrt{3}} \left(2J^2 - 3\{J^i, J_s^i\}\right) \\
&= \frac{1}{2\sqrt{3}} \left(-J^2 + 3I^2 - 3J_s^2\right), \\
\{J^i, G^{i3}\} &= \frac{1}{2} \left(\{J^i, J_u^i\} - \{J^i, J_d^i\}\right) \\
&= \frac{1}{2} \left(V^2 - U^2 + J_u^2 - J_d^2\right),
\end{aligned}
\tag{2.16}
$$

where I, U and V are the isotopic spins of $SU(3)$. The baryon states of physical interest are those with spin, isospin and strangeness of order unity. Thus, these baryons

have matrix elements of T^8, $\{J^i, G^{i8}\}$, T^3 and $\{J^i, G^{i3}\}$ which are $O(N_c)$, $O(1)$, $O(1)$ and $O(N_c)^\dagger$, respectively. The N_c-dependence of T^8 is trivial; the coefficient of the $O(N_c)$ piece in T^8 matrix elements is the same for all the physical baryons, so it cancels in any mass difference. Furthermore, consider any operator $\mathcal{O}_m/N_c^{m-1}$. Then the part of $\{T^8, \mathcal{O}_m\}/N_c^m$ which originates in the $O(N_c)$ piece of T^8 is of the form

$$(\text{const})N_c \times \mathcal{O}_m/N_c^m, \qquad (2.17)$$

and can be absorbed into the operator $\mathcal{O}_m$. We therefore do not need to worry about the $O(N_c)$ contribution to the matrix elements of T^8, and we conclude that for the special case $a = 8$, it is possible to truncate the expansion (2.13) after the first term, T^8, up to a correction of relative order $1/N_c$ arising from $\{J^i, G^{i8}\}/N_c$. The situation for $a = 3$ does not simplify since the matrix elements of T^3 and $\{J^i, G^{i3}\}/N_c$ are both order one. The first truncation of the expansion for $a = 3$ retains these two operators.

C. $(0, 27)$

The $1/N_c$ expansion for a $(0, 27)$ operator can be determined using the operator reduction rule of Ref. [12]. There are three two-body operators which transform as a flavor 27: spin-zero ($\{T^a, T^b\}$), spin-one ($\{G^{ia}, T^b\} + \{G^{ib}, T^a\}$) and spin-two ($\{G^{ia}, G^{jb}\} + \{G^{ib}, G^{ja}\}$). Spin-zero 27 operators can be obtained from the latter two by forming tensor products with factors of J to saturate spin indices. Thus, there is a unique two-body $(0, 27)$ operator,

$$\mathcal{O}_2^{ab} = \{T^a, T^b\}. \qquad (2.18)$$

There is one three-body $(0, 27)$ operator, which is the tensor product of the spin-one 27 two-body operator and J^i,

$$\begin{aligned}
\mathcal{O}_3^{ab} &= \{J^i, \{T^a, G^{ib}\}\} + \{J^i, \{T^b, G^{ia}\}\} \\
&= \{T^a, \{J^i, G^{ib}\}\} + \{T^b, \{J^i, G^{ia}\}\}, \qquad (2.19)
\end{aligned}$$

where the second equality follows since J^i and T^a commute. The tensor product of the spin-two 27 and the spin-two combination $\{J^i, J^j\}$ yields the four-body $(0, 27)$ operator,

$$\mathcal{O}_4^{ab} = \{\{J^i, J^j\}, \{G^{ia}, G^{jb}\}\}, \qquad (2.20)$$

where projection of the spin-two pieces of $\{J^i, J^j\}$ and $\{G^{ia}, G^{jb}\}$ in Eq. (2.20) is implied. It is also to be understood that the flavor singlet and octet components of

the above three operators are subtracted off, so that each of the three operators is truly a $(0, 27)$. Note in all these cases the symmetry under exchange of flavor indices, as required for flavor-27 operators.

In general, the complete set of linearly independent $(0, 27)$ operators consists of three operator series, namely the three operators $\mathcal{O}_2^{ab}$, $\mathcal{O}_3^{ab}$, $\mathcal{O}_4^{ab}$ and anticommutators of these operators with J^2. Note that this implies that there are two different n-body operators for n even, $n \geq 4$, since $\{J^2, \mathcal{O}_2^{ab}\}$ and $\mathcal{O}_4^{ab}$ are both four-body operators. Thus, the $1/N_c$ expansion for a $(0, 27)$ operator has the form

$$\mathcal{O}^{ab} = \sum_{n=2,3}^{N_c} c_n \frac{1}{N_c^{n-1}} \mathcal{O}_n^{ab} + \sum_{n=4,6}^{N_c-1} d_n \frac{1}{N_c^{n-1}} \mathcal{O}_n^{ab}, \quad (2.21)$$

where c_n and d_n are unknown coefficients, and the d_n operators are the series of operators generated from $\mathcal{O}_4^{ab}$. Since the matrix elements of J are order one, the expansion (2.21) can be truncated for arbitrary a after the first three operators ($\mathcal{O}_2^{ab}$, $\mathcal{O}_3^{ab}$, $\mathcal{O}_4^{ab}$) up to corrections of relative order $1/N_c^2$. For $N_c = 3$, one only retains the two operators $\mathcal{O}_2^{ab}$ and $\mathcal{O}_3^{ab}$. Truncation after $\mathcal{O}_3^{ab}$ is valid for any N_c up to a relative correction of order $1/N_c^2$ only if one restricts the set of baryons of interest to the physical baryons.

There are three different $(0, 27)$ operators which are relevant for the analysis of the baryon mass splittings: $\mathcal{O}^{88}$ with $I = 0$, $\mathcal{O}^{83} = \mathcal{O}^{38}$ with $I = 1$, and $\mathcal{O}^{33}$ with $I = 2^\ddagger$. For the special case $a = b = 8$, it is possible to truncate the expansion (2.21) after the first operator in the expansion, $\mathcal{O}_2^{88}$, up to a correction of relative order $1/N_c$ arising from $\mathcal{O}_3^{88}/N_c$, since the matrix elements of $\{J^i, G^{i8}\}$ are $O(1)$. The truncations for $\mathcal{O}^{83}$ and $\mathcal{O}^{33}$ cannot be simplified further.

D. $(0, 64)$

The $1/N_c$ expansion for a $(0, 64)$ operator begins with a single three-body operator,

$$\mathcal{O}_3^{abc} = \{T^a, \{T^b, T^c\}\}, \qquad (2.22)$$

where it is understood that the singlet, octet and 27 components of the above operator are to be subtracted off so that only the flavor 64 component remains. For $N_c = 3$, this is the single operator which enters the analysis. For arbitrary N_c, there are three additional operators,

$$\begin{aligned}
\mathcal{O}_4^{abc} &= \{T^a, \{T^b, \{J^i, G^{ic}\}\}\}, \\
\mathcal{O}_5^{abc} &= \{T^a, \{\{J^i, G^{ib}\}, \{J^i, G^{ic}\}\}\}, \qquad (2.23) \\
\mathcal{O}_6^{abc} &= \{\{J^i, G^{ia}\}, \{\{J^i, G^{ib}\}, \{J^i, G^{ic}\}\}\}.
\end{aligned}$$

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†From Figs. 3 and 4, it is clear that the physical baryons have J_u, J_d, U-spin and V-spin of $O(N_c)$. The $O(N_c^2)$ contribution to $\{J^i, G^{i3}\}$ cancels exactly, but the $O(N_c)$ piece does not.

‡Subtraction to obtain operators with a unique value of isospin is understood.

All other operators of the expansion are generated from the above four by anticommuting with J^2. Truncation of the $1/N_c$ expansion after the first operator $\mathcal{O}_3^{abc}$ is valid for general N_c up to a relative correction of order $1/N_c^2$ only if one restricts the set of baryons of interest to the physical baryons.

There are four different $(0,64)$ operators which are relevant for the analysis of the baryon mass splittings: $\mathcal{O}^{888}$ with $I=0$, $\mathcal{O}^{883}$ with $I=1$, $\mathcal{O}^{833}$ with $I=2$, and $\mathcal{O}^{333}$ with $I=3$.

E. $(0, 10 + \overline{10})$

The $1/N_c$ expansion for a $(0, 10 + \overline{10})$ operator begins with a single three-body operator,

$$\mathcal{O}_3^{ab} = \{T^a, \{J^i, G^{ib}\}\} - \{T^b, \{J^i, G^{ia}\}\}. \qquad (2.24)$$

For $N_c = 3$, this is the only operator which enters the analysis. For arbitrary N_c, there are additional operators generated by anticommutators of J^2 with the above operator. Thus, the truncation of the general $1/N_c$ expansion of a $(0, 10 + \overline{10})$ operator after this first operator is valid up to a relative correction of order $1/N_c^2$.

There is one $(0, 10 + \overline{10})$ operator which is relevant for the analysis of the baryon mass splittings: $\mathcal{O}^{38}$ with $I=1$.

III. COMPLETE SET OF OPERATORS

In this section, we present the complete set of operators which parametrize the baryon masses for $N_c = 3$. The set of operators given here are also the operators required for an analysis for arbitrary N_c when one restricts the baryon states of interest to the physical baryons. There are 19 Hermitian operators which span the space parametrized by the 18 baryon masses and the single off-diagonal mixing mass $\bar{\Lambda}\Sigma^0 = \bar{\Sigma}^0\Lambda$. Since $\bar{\Lambda}\Sigma^0 = \bar{\Sigma}^0\Lambda$ by hermiticity, we refer to the off-diagonal mass as $\Lambda\Sigma^0$ in this work[§]. The 19 operators that we require are obtained by truncating the general $1/N_c$ expansions derived in the previous section at three-body operators.

The baryon mass operator M can be written in terms of mass operators M_I^R, which belong to $SU(3)$ representations R with isospin I,

$$M = \sum_{R,I} M_I^R. \qquad (3.1)$$

For the restricted set of baryon states, the most general mass operator is given by

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[§]A $(\bar{\Lambda}\Sigma^0 - \bar{\Sigma}^0\Lambda)$ mass difference, corresponding to the representation $(0, 10-\overline{10})$, is disallowed by time reversal invariance.

$$M = M_0^1 + M_0^8 + M_0^{27} + M_0^{64} + M_1^{10+\overline{10}}$$
$$+ M_1^8 + M_1^{27} + M_1^{64} + M_2^{27} + M_2^{64} + M_3^{64} . \qquad (3.2)$$

Each of the mass operators M_I^R in Eq. (3.2) may be expanded in a $1/N_c$ expansion of the form

$$M_I^R = \sum_{n=0}^{3} c_{(n)}^{R,I} \frac{1}{N_c^{n-1}} \mathcal{O}_n^{R,I}, \qquad (3.3)$$

where the summation is over all n-body operators $\mathcal{O}_n^{R,I}$ with the same transformation properties under $SU(3)$ and isospin as M_I^R. The unknown operator coefficients are denoted by $c_{(n)}^{R,I}$. The explicit expansions for each of the representations appearing in Eq. (3.2) are obtained from the results of Sec. II and are given below. The operator expansions divide into the $I = 0$ expansions

$$M_0^1 = c_{(0)}^{1,0} N_c \mathbb{1} + c_{(2)}^{1,0} \frac{1}{N_c} J^2,$$

$$M_0^8 = c_{(1)}^{8,0} T^8 + c_{(2)}^{8,0} \frac{1}{N_c}\{J^i, G^{i8}\} + c_{(3)}^{8,0} \frac{1}{N_c^2}\{J^2, T^8\},$$

$$M_0^{27} = c_{(2)}^{27,0} \frac{1}{N_c}\{T^8, T^8\} + c_{(3)}^{27,0} \frac{1}{N_c^2}\{T^8, \{J^i, G^{i8}\}\},$$

$$M_0^{64} = c_{(3)}^{64,0} \frac{1}{N_c^2}\{T^8, \{T^8, T^8\}\}; \qquad (3.4)$$

the $I = 1$ expansions

$$M_1^8 = c_{(1)}^{8,1} T^3 + c_{(2)}^{8,1} \frac{1}{N_c}\{J^i, G^{i3}\} + c_{(3)}^{8,1} \frac{1}{N_c^2}\{J^2, T^3\},$$

$$M_1^{27} = c_{(2)}^{27,1} \frac{1}{N_c}\{T^3, T^8\}$$
$$+ c_{(3)}^{27,1} \frac{1}{N_c^2} \left(\{T^3, \{J^i, G^{i8}\}\} + \{T^8, \{J^i, G^{i3}\}\}\right),$$

$$M_1^{64} = c_{(3)}^{64,1} \frac{1}{N_c^2}\{T^8, \{T^8, T^3\}\}, \qquad (3.5)$$

$$M_1^{10+\overline{10}} = c_{(3)}^{10+\overline{10},1} \frac{1}{N_c^2} \left(\{T^3, \{J^i, G^{i8}\}\} - \{T^8, \{J^i, G^{i3}\}\}\right);$$

the $I = 2$ expansions

$$M_2^{27} = c_{(2)}^{27,2} \frac{1}{N_c}\{T^3, T^3\} + c_{(3)}^{27,2} \frac{1}{N_c^2}\{T^3, \{J^i, G^{i3}\}\},$$

$$M_2^{64} = c_{(3)}^{64,2} \frac{1}{N_c^2}\{T^8, \{T^3, T^3\}\}; \qquad (3.6)$$

and the $I = 3$ expansion

$$M_3^{64} = c_{(3)}^{64,3} \frac{1}{N_c^2}\{T^3, \{T^3, T^3\}\}. \qquad (3.7)$$

Note that, as in the previous section, the 19 operators appearing in Eqs. (3.4–3.7) are to be regarded as subtracted operators, so that each operator transforms according to the $SU(3)$ and isospin representations stated. Thus, the flavor 27 operators require subtraction of singlet and

octet components, the 64 operators require removal of singlet, octet and 27 components, and $10 + \overline{10}$ operator requires removal of an octet component. One further level of subtraction diagonalizes the operators into channels of unique isospin $I = 0, 1, 2, 3$.

It is easy to incorporate flavor symmetry breaking into the $1/N_c$ analysis by associating powers of symmetry-breaking parameters with each of the coefficients appearing in Eqs. (3.4–3.7). There are two sources of flavor symmetry breaking. The first source is the quark mass matrix, which introduces the perturbations

$$\epsilon \mathcal{H}^8 + \epsilon' \mathcal{H}^3, \qquad (3.8)$$

where ϵ is an $SU(3)$-violating parameter and ϵ' is an isospin-violating parameter. The magnitude of these symmetry-breaking parameters is governed by quark mass differences divided by the chiral symmetry breaking scale, which is of order one GeV. The symmetry-breaking parameter $\epsilon \sim 0.25$ is comparable to a $1/N_c$ effect in QCD, so we must carefully keep track of all powers of ϵ in the perturbative analysis. The isospin-breaking parameter satisfies $\epsilon' \ll \epsilon$. A typical isospin mass splitting is order several MeV, whereas the overall $O(N_c)$ mass of the baryons is about one GeV. Thus, ϵ' is comparable to an effect of order $1/N_c^5$ in QCD. The $1/N_c$ expansions we have constructed contain only a few orders of $1/N_c$, so we only need to consider isospin-breaking effects to linear order in the isospin-breaking parameter ϵ'. The second source of flavor symmetry breaking is the quark charge matrix. Electromagnetic mass splittings are second order in the quark charge matrix, and are suppressed by $\alpha_{em}/4\pi$. These splittings are typically of order a few MeV in magnitude, which is comparable to the isospin mass splittings arising from the quark mass matrix but is negligible compared to $SU(3)$ mass splittings. We introduce the symmetry-breaking parameter ϵ'' for the electromagnetic mass splittings, where $\epsilon'' \sim \epsilon'$. The electromagnetic effects can occur in the $I = 0, 1, 2$ channels; the $I = 0$ contribution can be neglected, and both the $I = 1$ and 2 contributions are suppressed by ϵ''. The symmetry-breaking parameters associated with the mass operators at leading order in flavor symmetry breaking are listed in Table I[**].

Finally, we describe in more detail the relation between the $1/N_c$ expansion when it is combined with a perturbative analysis in flavor symmetry breaking and a pure $SU(6)$ analysis. Not all of the mass operators of the $1/N_c$ expansion Eqs. (3.4–3.7) with perturbative flavor breaking transform according to unique $SU(6)$ representations. Thus, it is not possible to identify the n-body label $n = 0, 1, 2, 3$ of the coefficients $c_{(n)}^{R,I}$ with the 1,

35, 405 and 2695 dimensional representations of $SU(6)$, respectively. This subtlety occurs because some of the n-body operators written in Eqs. (3.4–3.7) are not pure n-body $SU(6)$ operators; the operators contain components which reduce by the identities of Ref. [12] to lower-body operators in different $SU(6)$ representations. An upper-triangular matrix summarizes the relation between the pure n-body $SU(6)$ (rows) and n-body flavor symmetry-breaking (columns) operator bases of the $1/N_c$ expansion:

$$\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 35 \\ 405 \\ 2695 \end{array}
\begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix},
\end{array} \qquad (3.9)$$

where $*$ indicates an entry which is not necessarily zero. From this matrix, one finds, for example, that 3-body flavor operators transform as $1 + 35 + 405 + 2695$, but that pure 3-body 2695 operators only appear in 3-body flavor operators.

IV. MASS RELATIONS

We now study the mass relations which can be obtained using the operator expansions of the previous section. In Table II, we compile the mass combinations associated with the neglect of each of the 19 operators of Sec. III, and with the irreducible representations of $SU(6)$ whenever the two do not coincide. The mass combinations are divided into isospin sectors $I = 0, 1, 2, 3$. The definitions of the baryon isospin mass combinations used in the table are presented in the subsections which follow. The mass combinations associated with the 19 operators in Eqs. (3.4–3.7) are labeled by their coefficients $c_{(n)}^{R,I}$. The $1/N_c$ suppressions and the flavor-breaking parameters associated with each mass combination are tabulated. The $1/N_c$ suppression factors assigned to each mass combination include the implicit N_c-dependence of operator matrix elements as well as the explicit $1/N_c$ factors displayed in Eq. (3.4–3.7). The flavor-breaking suppression factors are obtained from Table I. Mass combinations which also correspond to single $SU(6)$ representations appear with a check in the column with heading label $SU(6)$; otherwise a "No" appears. The $SU(6)$ mass combinations which differ from the above combinations are listed in the subsequent blocks of the table. These combinations are labeled by coefficients $c_D^{R,I}$ where the subscript denotes the dimension of the $SU(6)$ representation. The $1/N_c$ and the leading flavor-breaking suppressions for the $SU(6)$ combinations are listed. Note that the flavor suppressions for the $SU(6)$ combinations follow from the perturbative flavor-breaking operator analysis and are not consequences of the analysis in terms of operators with definite $SU(6)$ transformation properties.

We can understand which mass combinations in Table II coincide with $SU(6)$ mass combinations. The mass

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[**]Note that the true flavor suppression of $M_1^{10+\overline{10}}$ comes from terms of $O(e^2 m_s)$ [21] and is hence $\epsilon'' \epsilon$. In Table I we list the naive factor $\epsilon' \epsilon$, which is equivalent because $\epsilon'' \sim \epsilon'$.

combination associated with the neglect of each operator in the basis Eqs. (3.4–3.7) is broken by this operator alone, *i.e.* all of the other operators in the basis vanish on this mass combination. Recall (3.9) that the 3-body flavor operators are the only operators in the basis which contain components transforming according to the 2695 representation of $SU(6)$. Since all of the other $SU(6)$ representations occur in lower-body operators as well, the only mass combination which vanishes for all lower-body flavor operators are mass combinations in the 2695. Therefore, all of the 3-body mass combinations in the first block of Table II correspond to mass combinations in the 2695 representation of $SU(6)$. In addition, for the $SU(3)$ singlet expansion M_0^1, which involves only even-body operators, neglect of the 2-body operator J^2 results in a mass combination in the 405 representation of $SU(6)$. In general, neglect of the highest-body operator in any perturbative flavor symmetry-breaking expansion leads to a mass combination corresponding to a definite $SU(6)$ representation. None of the mass combinations following from the neglect of other n-body operators appearing in Eqs. (3.4–3.7) has this property.

The last column of Table II presents the experimental accuracy of mass relations obtained by setting each mass combination equal to zero. These percentage accuracies are determined by evaluating the given mass combination, and then dividing by one-half the sum of the absolute values of all of the terms in the same mass combination. Stated another way, the combination is organized as $lhs = rhs$, where lhs and rhs possess only baryon masses with positive coefficients, and then we compute $|(lhs - rhs)/((lhs + rhs)/2)|$. The purpose of this normalization is twofold: first, this expression is invariant under multiplication of all baryon coefficients by the same constant, and second, the resulting number is dimensionless. Because the denominator is a sum of baryon masses with the same size coefficients as in the numerator, the ratio represents a scale-invariant measure of how much the numerator mass difference is suppressed. The experimental accuracies listed in Table II are obtained using the measured baryon masses and mass differences of the Particle Data Group [22], although there is controversy over isospin splittings in the decuplet (see, *e.g.* [23]). The Δ^- mass is unmeasured; we eliminate this mass from all of the $I = 0, 1, 2$ Δ mass combinations using the sole $I = 3$ mass relation (see Subsec. D), which is satisfied to high orders in the $1/N_c$ expansion and flavor symmetry breaking. Nevertheless, large experimental uncertainties remain in the $I = 1$ and 2 isospin splittings of the Δ, and this prevents a meaningful comparison of the predicted theoretical hierarchy of many of the $I = 1$ and 2 mass relations with experiment.

The experimental accuracies in Table II are to be compared with the the combined $1/N_c$ and flavor symmetry-breaking suppressions of a given mass combination. Note that the singlet baryon mass proportional to the operator $\mathbb{1}$ is order N_c, so the $1/N_c$ suppression of each mass combination relative to this singlet mass contains one more power of $1/N_c$ than what is listed in the table. In the following four subsections, we analyze the mass relations arising in the $I = 0, 1, 2, 3$ channels, respectively.

A. $I = 0$ Mass Relations

There are eight linear combinations of the octet and decuplet masses which transform as $I = 0$: Λ, Ω, and

$$N_0 = \frac{1}{2}\left(p + n\right),$$

$$\Sigma_0 = \frac{1}{3}\left(\Sigma^+ + \Sigma^0 + \Sigma^-\right),$$

$$\Xi_0 = \frac{1}{2}\left(\Xi^0 + \Xi^-\right),$$

$$\Delta_0 = \frac{1}{4}\left(\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-\right), \qquad (4.1)$$

$$\Sigma_0^* = \frac{1}{3}\left(\Sigma^{*+} + \Sigma^{*0} + \Sigma^{*-}\right),$$

$$\Xi_0^* = \frac{1}{2}\left(\Xi^{*0} + \Xi^{*-}\right).$$

The eight $I = 0$ mass combinations in the first block of Table II correspond to the eight operators in Eq. (3.4). Each mass combination is assigned definite $1/N_c$ and $SU(3)$ flavor symmetry-breaking suppression factors. The $SU(3)$ flavor symmetry-breaking assignments are easy to understand: the octet combinations arise at first order in ϵ; the 27 combinations at order ϵ^2; and the 64 at order ϵ^3. The experimental accuracies in the last column of Table II allow us to compare the relative suppressions of each operator in the $1/N_c$ expansion; values are given for all of the mass combinations except the singlet combination $c_{(0)}^{1,0}$, which is not suppressed[††]. The experimental accuracies exhibit the hierarchy predicted by the combined $1/N_c$ and flavor symmetry-breaking suppressions; mass combinations which are more highly suppressed in the $1/N_c$ and ϵ expansions correspond to proportionally smaller numbers in Table II. In addition, comparison of the experimental accuracies of the mass combinations $c_{(2)}^{1,0}$ and $c_{(1)}^{8,0}$ indicates that the flavor symmetry-breaking parameter ϵ is indeed comparable to one factor of $1/N_c$.

$I = 0$ mass relations are obtained by successively neglecting operators in the mass expansion. The most highly suppressed operator is the unique 64 operator occurring at order ϵ^3/N_c^2 in the combined $1/N_c$ and flavor-breaking expansions. Neglect of this operator yields the mass relation

$$\frac{1}{2}(\Delta_0 - 3\Sigma_0^* + 3\Xi_0^* - \Omega) = 0, \qquad (4.2)$$

[††]The experimental accuracy for this mass combination is $O(1)$, as expected.

with an expected relative accuracy of $\epsilon^3/N_c^{3\ddagger\ddagger}$. The naive estimate of the quantity ϵ^3/N_c^3 is consistent with the experimental value $0.09 \pm 0.03\%$. At next subleading order, $O(\epsilon^2/N_c^2)$, the operator $c_{(3)}^{27,0}$ is neglected, resulting in the mass relation

$$2\left(\frac{3}{4}\Lambda + \frac{1}{4}\Sigma_0 - \frac{1}{2}(N_0 + \Xi_0)\right) =$$
$$-\frac{1}{7}(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega). \qquad (4.3)$$

Note that the left-hand side of Eq. (4.3) is proportional to the Gell-Mann–Okubo formula, whereas the right-hand side is one linear combination of the two decuplet equal spacing rules (Eq. (4.2) is the other). The predicted ϵ^2/N_c^3 accuracy of relation (4.3) is consistent with the experimental value. To next order in the combined $1/N_c$ and ϵ expansions, one neglects the two mass operators $c_{(3)}^{8,0}$ and $c_{(2)}^{27,0}$ occurring at the comparable orders $O(\epsilon/N_c^2)$ and $O(\epsilon^2/N_c)$, respectively. The two mass relations listed in Table II are satisfied at the expected level of accuracy. Note that mass relation $c_{(2)}^{27,0}$ is satisfied to a somewhat greater level of accuracy than mass relation $c_{(3)}^{8,0}$, although both are consistent with the canonical values from the combined ϵ and $1/N_c$ expansions with natural-size coefficients. By neglecting the $c_{(2)}^{8,0}$ operator, one obtains another mass relation at $O(\epsilon/N_c)$, with canonical accuracy of 3% as compared to an experimental value of 6%. The two final mass combinations $c_{(1)}^{8,0}$ and $c_{(2)}^{1,0}$ arise at the comparable orders $1/N_c$ and ϵ, respectively, so neither operator can be neglected relative to the other, and no additional mass relation is obtained. It is valid, however, to neglect the $c_{(2)}^{1,0}$ operator relative to the $c_{(0)}^{1,0}$ operator when considering the $1/N_c$ expansion in the singlet channel. The mass relation following from the neglect of the $c_{(2)}^{1,0}$ operator J^2/N_c in the singlet mass expansion is the trivial relation which equates the average octet and decuplet masses,

$$\frac{1}{8}(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) =$$
$$\frac{1}{10}(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega). \qquad (4.4)$$

The predicted accuracy of this relation, which is a measure of the size of decuplet-octet mass splittings, is of order

$$\frac{1}{N_c^2}\left((J^2)_{3/2} - (J^2)_{1/2}\right) = \frac{1}{N_c^2}\left(\frac{15}{4} - \frac{3}{4}\right) = \frac{3}{N_c^2}, \quad (4.5)$$

which is consistent with the experimental value of 18%.

$\ddagger\ddagger$This relation is also satisfied by one-loop diagrams in chiral perturbation theory [24].

Finally, we consider how our results relate to previous analyses. It is possible to analyze the $I = 0$ baryon masses in a $1/N_c$ expansion alone, in contrast to the combined $1/N_c$ and flavor symmetry-breaking expansion of this work. In the $1/N_c$ expansion, the most highly suppressed $I = 0$ operators are $O(1/N_c^2)$, namely the operators with coefficients $c_{(3)}^{8,0}$, $c_{(3)}^{27,0}$ and $c_{(3)}^{64,0}$. Neglect of these operators yields the three mass relations in Table II corresponding to these three coefficients. An equivalent set of mass relations is

$$\Sigma_0^* - \Sigma_0 = \Xi_0^* - \Xi_0 \qquad (0.88 \pm 0.02\%),$$
$$\frac{3}{4}\Lambda + \frac{1}{4}\Sigma_0 - \frac{1}{2}(N_0 + \Xi_0)$$
$$= -\frac{1}{4}(\Omega - \Sigma_0^* - \Xi_0^* + \Delta_0) \qquad (0.16 \pm 0.01\%), \quad (4.6)$$
$$\frac{1}{2}(\Sigma_0^* - \Delta_0) - (\Xi_0^* - \Sigma_0^*) + \frac{1}{2}(\Omega - \Xi_0^*) = 0,$$

where the first relation is a linear combination of the $c_{(3)}^{8,0}$, $c_{(3)}^{27,0}$ and $c_{(3)}^{64,0}$ mass relations; the second is a linear combination of the $c_{(3)}^{27,0}$ and $c_{(3)}^{64,0}$ relations; and the third is the $c_{(3)}^{64,0}$ mass relation (4.2). Eq. (4.6) is the set of mass relations obtained in Refs. [4,12] using $SU(2) \times U(1)$ symmetry. Since the $c_{(3)}^{8,0}$, $c_{(3)}^{27,0}$ and $c_{(3)}^{64,0}$ operators are the $I = 0$ operators associated with the 2695 of $SU(6)$, these relations also were obtained in the $SU(6)$ $I = 0$ analysis of Ref. [20]. The experimental accuracies of these relations clearly exhibit the hierarchy $\epsilon/N_c^2 : \epsilon^2/N_c^2 : \epsilon^3/N_c^2$, so there is evidence for both the $1/N_c$ and flavor-breaking suppressions. In a second analysis, Ref. [12] obtained $I = 0$ mass relations valid at linear order in the flavor symmetry breaking ϵ and neglecting operators suppressed by $1/N_c^2$. This analysis corresponds to the neglect of the $c_{(3)}^{8,0}$, $c_{(2)}^{27,0}$, $c_{(3)}^{27,0}$ and $c_{(3)}^{64,0}$ operators, all of which are suppressed by $1/N_c^2$ and/or ϵ^2. The four resulting mass relations (see Eqs. (10.11)-(10.14) of Ref. [12]) are the Gell-Mann–Okubo formula, the two decuplet equal spacing rules, and $\Sigma_0^* - \Sigma_0 = \Xi_0^* - \Xi_0$. In our analysis, the same set is obtained by truncating at order $\epsilon/N_c^2 \sim \epsilon^2/N_c$ in the combined $1/N_c$ and flavor symmetry-breaking expansions. The analysis of this work, however, exhibits the complete $1/N_c$ and symmetry-breaking structure of these relations, and obtains the one additional relation at order ϵ/N_c. From our analysis, we are able to conclude that the $1/N_c$ and flavor symmetry-breaking suppression factors are both required to describe the observed hierarchy of the $I = 0$ sector.

B. $I = 1$ Mass Relations

There are seven $I = 1$ mass combinations: six $I = 1$ mass splittings

$$N_1 = (p - n),$$

$$\Sigma_1 = (\Sigma^+ - \Sigma^-),$$
$$\Xi_1 = (\Xi^0 - \Xi^-),$$
$$\Delta_1 = (3\Delta^{++} + \Delta^+ - \Delta^0 - 3\Delta^-), \qquad (4.7)$$
$$\Sigma_1^* = (\Sigma^{*+} - \Sigma^{*-}),$$
$$\Xi_1^* = (\Xi^{*0} - \Xi^{*-}),$$

and one off-diagonal mass $\Lambda\Sigma^0$.

The most highly suppressed $I = 1$ operator in the combined $1/N_c$ and flavor-breaking expansions is the unique 64 operator occurring at order $\epsilon'\epsilon^2/N_c^2$. Neglect of this operator yields the mass relation

$$\Delta_1 - 10\Sigma_1^* + 10\Xi_1^* = 0, \qquad (4.8)$$

with a predicted accuracy of $\epsilon'\epsilon^2/N_c^3 \approx 10^{-5}$. A meaningful comparison of this suppression factor with experiment is not possible at present because of large experimental uncertainties in decuplet $I = 1$ splittings, particularly Δ_1. At next subleading order in the combined $1/N_c$ and flavor symmetry-breaking expansions, one neglects the four operators $c_{(3)}^{8,1}$, $c_{(2)}^{27,1}$, $c_{(3)}^{27,1}$ and $c_{(3)}^{10+\overline{10},1}$ occurring at comparable orders ϵ'/N_c^2 and $\epsilon'\epsilon/N_c$. The $c_{(3)}^{10+\overline{10},1}$ mass relation is the Coleman–Glashow relation

$$N_1 - \Sigma_1 + \Xi_1 = 0, \qquad (4.9)$$

which is known to be very accurate; the experimental accuracy for this relation is consistent with zero. The central value, however, is completely in line with the naive estimate of the quantity $\epsilon'\epsilon/N_c^2$, where ϵ' is numerically about $1/N_c^5$ in QCD. More accurate measurements of Σ_1 and Ξ_1 are required to fully test this relation at the level predicted by our analysis. Notice that the two combinations $c_{(2)}^{27,1}$ and $c_{(3)}^{27,1}$ are exactly the same order in $1/N_c$ and flavor breaking, and exhibit the same combinations of octet and decuplet masses. Thus, an equivalent, simpler pair of mass relations at order $\epsilon'\epsilon/N_c$ is

$$N_1 - \Xi_1 + 2\sqrt{3}\Lambda\Sigma^0 = 0, \qquad (4.10)$$
$$\Delta_1 - 3\Sigma_1^* - 4\Xi_1^* = 0. \qquad (4.11)$$

Relation (4.11) is satisfied to an experimental accuracy of $0.27\pm0.10\%$. Again, the large uncertainty in Δ_1 prevents a meaningful comparison of this value with the theoretical accuracy of $\epsilon'\epsilon/N_c^2$. The other two relations at this order in the expansion, $c_{(3)}^{8,1}$ and Eq. (4.10), involve the unmeasured $\Lambda\Sigma^0$ mass, and cannot be compared with experiment. Finally, the two remaining mass combinations $c_{(1)}^{8,1}$ and $c_{(2)}^{8,1}$ both appear at order ϵ' in the combined $1/N_c$ and flavor symmetry-breaking expansions, so neither operator can be neglected relative to the other, and no additional mass relation is obtained.

Comparison of our $I = 1$ mass hierarchy with experiment is limited by the large experimental uncertainty in the splitting Δ_1 and the presence of the unknown parameter $\Lambda\Sigma^0$. It is possible to extract additional information

about the $I = 1$ mass hierarchy by eliminating these uncertain parameters. One may add to a given mass combination any other combinations which are of the same or higher order in the combined flavor and $1/N_c$ expansions. Such a linear combination remains at the same order in the combined expansions as the original one, although it no longer necessarily corresponds to a single $SU(3)$ representation. We use the $c_{(3)}^{64,1}$ mass relation Eq. (4.8) to eliminate Δ_1 from all other $I = 1$ mass combinations, and Eq. (4.10) to eliminate $\Lambda\Sigma^0$. In its own right, this equation predicts

$$\Lambda\Sigma^0 = -1.47 \pm 0.17 \text{ MeV}. \qquad (4.12)$$

We now analyze the results of these substitutions. The $c_{(3)}^{10+\overline{10},1}$ mass combination does not involve either Δ_1 or $\Lambda\Sigma^0$, and so is unaffected by this procedure. With these substitutions, the four remaining mass combinations $c_{(1)}^{8,1}$, $c_{(2)}^{8,1}$, $c_{(3)}^{8,1}$ and Eq. (4.11) reduce respectively to the four mass combinations given in the third block of the $I = 1$ sector of Table II. Note that the $1/N_c$ and flavor symmetry-breaking assignments of these combinations are identical to those of the original $SU(3)$ combinations. The assignments for the third combination do not appear in the table; it is the linear combination of $c_{(3)}^{8,1}$ and Eq. (4.10) which eliminates $\Lambda\Sigma^0$, and so combines order $\epsilon'\epsilon/N_c$ and ϵ'/N_c^2 contributions, which are comparable in the combined $1/N_c$ and flavor-breaking expansions.

From experimental values for these four mass combinations, we conclude that our predicted flavor-breaking and $1/N_c$ hierarchy is also evident in the $I = 1$ splittings. The first two combinations are expected to work at the level ϵ'/N_c, and their experimental accuracies are similar and consistent with $\epsilon' \approx 1/N_c^5$. The last relation

$$\Sigma_1^* = 2\Xi_1^* \qquad (4.13)$$

has an expected accuracy of $\epsilon'\epsilon/N_c^2$. The central value of the experimental accuracy is consistent with a suppression of ϵ/N_c relative to the first two mass combinations. The error on this experimental accuracy is large, however. The central value of the experimental accuracy of the third relation is surprisingly small, but its uncertainty puts it into the same range as that of the fourth relation. Likewise, the central value of the experimental accuracy for the $c_{(3)}^{10+\overline{10},1}$ relation lies within this same range. However, none of these last three relations is measured accurately enough to test our predicted hierarchy conclusively. Reasonable improvements in the measurement of $I = 1$ mass splittings would enable a substantive comparison.

C. $I = 2$ Mass Relations

There are three $I = 2$ splittings:

$$\Sigma_2 = (\Sigma^+ - 2\Sigma^0 + \Sigma^-),$$
$$\Delta_2 = (\Delta^{++} - \Delta^+ - \Delta^0 + \Delta^-), \qquad (4.14)$$
$$\Sigma_2^* = (\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}).$$

The most highly suppressed $I = 2$ operator in the combined $1/N_c$ and flavor-breaking expansions is the unique 64 operator occurring at order $\epsilon''\epsilon/N_c^2$. Neglect of this operator yields the mass relation

$$\Delta_2 = 2\Sigma_2^*, \qquad (4.15)$$

with a predicted accuracy of $\epsilon''\epsilon/N_c^3 \approx 3 \times 10^{-5}$. A meaningful comparison of this suppression factor with experiment is not possible because of large experimental uncertainties in the decuplet $I = 2$ splittings. The two remaining mass combinations $c_{(2)}^{27,2}$ and $c_{(3)}^{27,2}$ are both order $\epsilon''\epsilon/N_c$, so neither operator can be neglected relative to the other, and no additional mass relation is obtained.

The hierarchy of $I = 2$ mass combinations is completely consistent with the predictions of the combined $1/N_c$ and flavor-breaking expansions. Recall that the $I = 2$ flavor symmetry-breaking parameter ϵ'' is comparable to the $I = 1$ parameter ϵ'. Notice from Table II that all $I = 2$ combinations (and hence any linear combination of them) are suppressed by one factor of $1/N_c$ relative to the largest $I = 1$ combinations. The experimental accuracy of the $c_{(2)}^{27,2}$ combination is suppressed at this level relative to the two measured $O(\epsilon')$ $I = 1$ mass combinations in Table II. The $c_{(3)}^{27,2}$ combination may also be suppressed at this level, but its experimental accuracy is too poorly known. In addition, these two $I = 2$ combinations are predicted to be suppressed by $\epsilon'' \approx 5 \times 10^{-3}$ relative to the largest $I = 0$ mass combination $c_2^{1,0}$, and consistency with this prediction is also borne out in Table II. Unfortunately, however, the uncertainties of the experimental accuracies for the $I = 2$ mass combinations are substantial compared to their central values, so one cannot draw definitive conclusions about the accuracy of the observed hierarchy from the $I = 2$ masses.

D. $I = 3$ Mass Relations

The lowest-lying baryons admit only one $I = 3$ splitting,

$$\Delta_3 = (\Delta^{++} - 3\Delta^+ + 3\Delta^0 - \Delta^-). \qquad (4.16)$$

This mass combination corresponds to the single $I = 3$ operator, with coefficient $c_{(3)}^{64,3}$, that arises at order $\epsilon''\epsilon'/N_c^2$ in the combined $1/N_c$ and flavor-breaking expansions. Neglecting the $c_{(3)}^{64,3}$ operator yields the mass relation

$$\Delta_3 = 0. \qquad (4.17)$$

The suppression factor $\epsilon''\epsilon'/N_c^3$ is second order in the isospin-breaking parameters, and so is much smaller than

any of the other suppression factor in our analysis. Thus, we expect violations of Eq. (4.17) to be quite small. A naive estimate of the size of Δ_3 gives of order 10^{-3} MeV at most. A calculation [23] of this quantity in chiral perturbation theory, including loop effects, does not alter this conclusion. We have used the extreme accuracy of the mass relation (4.17) to eliminate the unknown Δ^- mass in the $I = 0, 1, 2$ Δ mass splittings.

V. COMPLETELY BROKEN $SU(3)$ SYMMETRY

The analysis of the $I = 0, 1, 2, 3$ baryon mass splittings can be performed using only $SU(2) \times U(1)$ flavor symmetry. Such an analysis yields mass relations which are valid to all orders in $SU(3)$ symmetry breaking. In this section, we reanalyze the baryon isospin mass splittings using $SU(2) \times U(1)$ flavor symmetry, treating isospin breaking as a small perturbation. The relevant spin-flavor symmetry group is $SU(4) \times SU(2) \times U(1)$, where the $SU(4)$ factor is the spin-flavor group of the two light flavors u and d; the $SU(2)$ factor is strange quark spin; and the $U(1)$ factor is the number of strange quarks. The analysis of the $I = 0$ sector was performed in Refs. [4,12], so we restrict the analysis here to $I = 1, 2, 3$ mass combinations.

The $SU(2) \times U(1)$ operator analysis uses one-body operators with definite isospin and strangeness instead of operators with definite $SU(3)$ transformation properties. In particular, this implies that the one-body operators $J_{ud}^i = J_u^i + J_d^i$ and J_s^i are used instead of J^i and G^{i8}, and that the strange quark number operator N_s is used instead of T^8. An $I \neq 0$ operator has the generic form

$$N_c \left(\frac{I^3}{N_c} \right)^p \left(\frac{1}{N_c^2} \{J_s^i, G^{i3}\} \right)^q, \qquad (5.1)$$

where $I = p + q$, times polynomials in N_s/N_c, I^2/N_c^2 and J^2/N_c^2 (see Ref. [12]). In analogy with Sec. II, we will restrict our analysis to the eleven isospin mass splittings of the physical baryons. For $N_c > 3$, we identify the physical baryon states with states at the top of the weight diagrams Figs. 3 and 4 that have strangeness, isospin, and spins J, J_{ud} and J_s of order one. The operator $\{J_s^i, G^{i3}\}$ has non-trivial matrix elements of order N_c for the physical baryons, so the matrix elements of I^3 and $\{J_s^i, G^{i3}\}/N_c$ are both $O(1)$ for these states. When we restrict our set of baryon states to the physical baryons, we only need to retain operators up to 3-body operators; 4- or higher-body operators are either redundant or vanish on this set. For $N_c > 3$, we find a total of eleven independent 0-, 1-, 2-, and 3-body operators. These same eleven operators also result from immediate specialization to $N_c = 3$.

The eleven $I \neq 0$ operators which span the $I = 1, 2, 3$ isospin splittings of the baryon octet and decuplet are generated using Eq. (5.1) and truncating at 3-body operators. In the following, we treat isospin symmetry break-

ing perturbatively. Because the isospin-breaking parameters ϵ' and ϵ'' are both comparable to effects of order $1/N_c^5$, it is sufficient to work to linear order in these parameters. There are two operators

$$c_{(1)}^1 I^3 + c_{(2)}^1 \frac{1}{N_c}\{J_s^i, G^{i3}\}, \qquad (5.2)$$

at leading order, $O(\epsilon')$, in the combined $1/N_c$ and isospin-breaking expansions, and four operators at next-to-leading order, $O(\epsilon'/N_c)$ or $O(\epsilon''/N_c)$,

$$d_{(2)}^1 \frac{N_s}{N_c} I^3 + d_{(3)}^1 \frac{N_s}{N_c^2}\{J_s^i, G^{i3}\}$$
$$+ d_{(2)}^2 \frac{1}{N_c}\{I^3, I^3\} + d_{(3)}^2 \frac{1}{N_c^2}\{I^3, \{J_s^i, G^{i3}\}\}. \qquad (5.3)$$

The operators in Eqs. (5.2) and (5.3) are understood to possess a single value of isospin as indicated by the coefficient superscripts; the subtraction of smaller isospin representations than the one indicated is implicit. The $I = 2$ operators appearing at next-to-leading order originate in electromagnetic interactions, as discussed in Sec. III, with comparable coefficients ($O(\epsilon'')$) to those of $I = 1$ operators ($O(\epsilon')$). Note that there is one additional operator which must be included for $N_c > 3$ if one does not restrict to physical baryons, namely the 4-body operator

$$\frac{1}{N_c^3}\{\{J_s^i, G^{i3}\}, \{J_s^i, G^{i3}\}\}. \qquad (5.4)$$

Finally, there are four additional 3-body operators at $O(\epsilon'/N_c^2)$ or $O(\epsilon''/N_c^2)$, $I^2 I^3$, $J^2 I^3$, $N_s^2 I^3$ and $N_s\{I^3, I^3\}$, and one additional 3-body operator $\{I^3, \{I^3, I^3\}\}$ at $O(\epsilon''\epsilon'/N_c^2)$. This last operator is second order in isospin breaking parameters. Note that there are additional higher-body operators at these orders which must be included if the set of baryon states is not restricted to the physical baryons.

Mass relations are obtained by successive neglect of $1/N_c$ suppressed operators. At leading order in the combined $1/N_c$ and isospin breaking expansions, one retains the two operators Eq. (5.2), which implies five mass relations amongst the seven $I = 1$ combinations. One obtains this same number of mass relations in the perturbative $SU(3)$ analysis of Sec. IV, but the spaces spanned by these two sets of relations are not equivalent, because the operator bases are not exactly the same. Specifically,

$$\{J^i, G^{i3}\} = \{J_{ud}^i, G^{i3}\} + \{J_s^i, G^{i3}\}, \qquad (5.5)$$

where

$$\{J_{ud}^i, G^{i3}\} = \frac{1}{2}(N_c - N_s + 2)I^3 \qquad (5.6)$$

by the operator identities [12], so the $SU(2) \times U(1)$ case introduces a higher-order piece (the $N_s I^3$) than present at the same order in the $SU(3)$ case. Nevertheless, four

of the five relations coincide; they may be written as the combinations $c_{(3)}^{8,1}$, $c_{(3)}^{27,1}$, $c_{(3)}^{10+\overline{10},1}$ and $c_{(3)}^{64,1}$ from Table II (the combination $c_{(2)}^{27,1}$ is broken by Eq. (5.2)). However, there is no reason to single out combinations corresponding to unique $SU(3)$ representations in the completely broken $SU(3)$ analysis. We instead choose linearly independent combinations with the smallest possible experimental uncertainties, so that one obtains the most stringent test of the $1/N_c$ hierarchy. From Eq. (5.3), one sees that two new $I = 1$ operators appear at next-to-leading order. Thus, two of the five $O(1)$ relations are broken at $O(1/N_c)$; we choose them to be the Coleman–Glashow relation (4.9) and the combination

$$3N_1 - \Xi_1 - 2\Xi_1^* = 0 \qquad (0.12 \pm 0.02\%), \qquad (5.7)$$

where the number given is the experimental accuracy of the relation as defined in Sec. IV. Furthermore, since the two leading-order operators (5.2) are $I = 1$, no $I = 2$ or $I = 3$ splittings are produced at $O(1)$ in the $1/N_c$ expansion.

At next-to-leading order (5.3), two new $I = 1$ operators appear, so we expect three $I = 1$ relations to remain. This counting seems to be at odds with the $1/N_c$ factors given in Table II for the perturbative $SU(3)$ case. In the $SU(3)$ analysis, one forms symmetric and antisymmetric combinations of an $O(1/N_c)$ and an $O(1/N_c^2)$ operator to obtain the pure $SU(3)$ 27 and $10 + \overline{10}$ representations, so that one finds instead two $O(1/N_c)$ combinations. In the present analysis, these two operators remain unmixed. We find that none of the combinations associated with the pure $SU(3)$ representations survive at this order (because $SU(3)$ is completely broken, this result is perhaps not surprising). A convenient choice for the set of three $I = 1$ relations at $O(1/N_c)$ is

$$10N_1 = \Delta_1 \qquad (0.19 \pm 0.10\%), \qquad (5.8)$$
$$-3N_1 + (\Sigma_1 - \Xi_1) + 2(\Sigma_1^* - \Xi_1^*) = 0$$
$$(0.002 \pm 0.016\%), \qquad (5.9)$$
$$2(\Sigma_1 - \Sigma_1^*) - (\Xi_1 - \Xi_1^*) = 2\sqrt{3}\Lambda\Sigma^0. \qquad (5.10)$$

The final relation predicts

$$\Lambda\Sigma^0 = -1.20 \pm 0.43 \text{ MeV}. \qquad (5.11)$$

In addition to the $I = 1$ operators, two $I = 2$ operators appear at $O(1/N_c)$, so there is one $I = 2$ relation at this order,

$$2\Sigma_2 - 3\Delta_2 + 4\Sigma_2^* = 0 \qquad (0.20 \pm 0.08\%), \qquad (5.12)$$

and one $I = 3$ relation (4.17). All of these relations are violated at order $1/N_c^2$; the $I = 3$ relation also requires an additional factor of isospin breaking.

The immediate conclusion we obtain from the analysis of this section is that the $SU(2) \times U(1)$ analysis does not produce as good a hierarchy as the perturbative $SU(3)$ analysis of the previous section. If we believe

only in completely broken $SU(3)$, then the $O(1)$ $I = 1$ relations Eqs. (4.9) and (5.7) should display accuracies of $O(\epsilon'/N_c) \approx 0.15\%$, and the $O(1/N_c)$ $I = 1$ relations Eqs. (5.8)–(5.10) accuracies of $O(\epsilon'/N_c^2) \approx 0.05\%$. In the $SU(2) \times U(1)$ analysis, we cannot explain why Eqs. (4.9) and (5.9) work so well experimentally. This fact provides additional evidence for the perturbative $SU(3)$ flavor-breaking analysis of Sec. IV: not only are the perturbative results consistent with experiment, but the accuracy of some mass relations cannot be explained otherwise. This conclusion is most obvious in the $I = 0$ sector; the analysis of this section shows that there is also evidence for it in the $I = 1$ sector.

VI. CONCLUSIONS

In summary, we conclude that there is striking evidence for the mass hierarchy predicted by a combined expansion in $1/N_c$ and $SU(3)$ flavor symmetry breaking, with flavor breaking treated perturbatively. Neither a $1/N_c$ nor a flavor expansion alone explains the observed hierarchy. In addition, a $1/N_c$ expansion treating only isospin breaking perturbatively fails to explain the hierarchy of the $I = 0$ and $I = 1$ mass combinations, so it is clearly better to treat $SU(3)$ as a perturbatively broken, rather than completely broken, symmetry.

Our analysis explicitly shows that the combined expansion differs from the old non-relativistic $SU(6)$ analysis, which neglected only mass combinations in the 2695. In the $1/N_c$ expansion, 2695 combinations are usually suppressed by a factor of $1/N_c^2$, which accounts for the fact that many of the $1/N_c$ mass relations coincide with $SU(6)$ combinations. There are additional relations obtained in the $1/N_c$ expansion satisfied at this same level of accuracy, however, which are not members of the 2695 and are therefore missed in the old $SU(6)$ analysis.

Finally, it is important to emphasize that improved measurements of baryon mass splittings (particularly decuplet isospin splittings) are needed to test a number of our mass relations at the level of accuracy predicted by the combined expansion. Even a modest decrease of experimental uncertainties in some mass combinations would be enough to permit one to distinguish conclusively between the predictions of this method and those of other possible hierarchies.

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TABLE I. Flavor symmetry breaking of M_I^R

M_0^1	M_0^8	M_0^{27}	M_0^{64}	M_1^8	M_1^{27}	$M_1^{10+\overline{10}}$	M_1^{64}	M_2^{27}	M_2^{64}	M_3^{64}
1	ϵ	ϵ^2	ϵ^3	ϵ'	$\epsilon'\epsilon$	$\epsilon'\epsilon$	$\epsilon'\epsilon^2$	ϵ''	$\epsilon''\epsilon$	$\epsilon''\epsilon'$

TABLE II. Mass Combinations: The isospin mass combinations appearing in the table are defined in Sec. IV. Orders of $1/N_c$ and flavor symmetry breaking are given for each combination. Experimental accuracies appear in the final column.

		$I=0$	$1/N_c$	Flavor	SU(6)	Expt.
$c_{(0)}^{1,0}$	$25(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) - 4(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$		N_c	1	No	$*$
$c_{(2)}^{1,0}$	$5(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) - 4(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$		$1/N_c$	1	$\checkmark$	$18.21 \pm 0.03\%$
$c_{(1)}^{8,0}$	$5(6N_0 - 3\Sigma_0 + \Lambda - 4\Xi_0) - 2(2\Delta_0 - \Xi_0^* - \Omega)$		1	ϵ	No	$20.21 \pm 0.02\%$
$c_{(2)}^{8,0}$	$N_0 - 3\Sigma_0 + \Lambda + \Xi_0$		$1/N_c$	ϵ	No	$5.94 \pm 0.01\%$
$c_{(3)}^{8,0}$	$(-2N_0 - 9\Sigma_0 + 3\Lambda + 8\Xi_0) + 2(2\Delta_0 - \Xi_0^* - \Omega)$		$1/N_c^2$	ϵ	$\checkmark$	$1.11 \pm 0.02\%$
$c_{(2)}^{27,0}$	$35(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 4(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$		$1/N_c$	ϵ^2	No	$0.37 \pm 0.01\%$
$c_{(3)}^{27,0}$	$7(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 2(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$		$1/N_c^2$	ϵ^2	$\checkmark$	$0.17 \pm 0.02\%$
$c_{(3)}^{64,0}$	$\Delta_0 - 3\Sigma_0^* + 3\Xi_0^* - \Omega$		$1/N_c^2$	ϵ^3	$\checkmark$	$0.09 \pm 0.03\%$
$c_1^{1,0}$	$(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) + 2(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$		N_c	1		$*$
$c_{35}^{8,0}$	$(N_0 - \Xi_0) + 2(2\Delta_0 - \Xi_0^* - \Omega)$		1	ϵ		$27.44 \pm 0.04\%$
$c_{405}^{8,0}$	$(7N_0 - 6\Sigma_0 + 2\Lambda - 3\Xi_0) - 2(2\Delta_0 - \Xi_0^* - \Omega)$		$1/N_c$	ϵ		$5.27 \pm 0.02\%$
$c_{405}^{27,0}$	$(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) + 4(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$		$1/N_c$	ϵ^2		$0.48 \pm 0.03\%$
		$I=1$				
$c_{(1)}^{8,1}$	$5(N_1 + 5\Sigma_1 + 4\Xi_1 + 2\sqrt{3}\,\Lambda\Sigma^0) - (\Delta_1 + 2\Sigma_1^* + \Xi_1^*)$		1	ϵ'	No	—
$c_{(2)}^{8,1}$	$-3N_1 + 3\Xi_1 + 4\sqrt{3}\,\Lambda\Sigma^0$		1	ϵ'	No	—
$c_{(3)}^{8,1}$	$(-7N_1 - 5\Sigma_1 + 2\Xi_1 + 6\sqrt{3}\,\Lambda\Sigma^0) + (\Delta_1 + 2\Sigma_1^* + \Xi_1^*)$		$1/N_c^2$	ϵ'	$\checkmark$	—
$c_{(2)}^{27,1}$	$35(N_1 - \Xi_1 + 2\sqrt{3}\,\Lambda\Sigma^0) - 2(\Delta_1 - 3\Sigma_1^* - 4\Xi_1^*)$		$1/N_c$	$\epsilon'\epsilon$	No	—
$c_{(3)}^{27,1}$	$7(N_1 - \Xi_1 + 2\sqrt{3}\,\Lambda\Sigma^0) - (\Delta_1 - 3\Sigma_1^* - 4\Xi_1^*)$		$1/N_c$	$\epsilon'\epsilon$	$\checkmark$	—
$c_{(3)}^{10+\overline{10},1}$	$N_1 - \Sigma_1 + \Xi_1$		$1/N_c$	$\epsilon'\epsilon$	$\checkmark$	$0.01 \pm 0.02\%$
$c_{(3)}^{64,1}$	$\Delta_1 - 10\Sigma_1^* + 10\Xi_1^*$		$1/N_c^2$	$\epsilon'\epsilon^2$	$\checkmark$	$0.08 \pm 0.05\%$
$c_{35}^{8,1}$	$(N_1 + 2\Sigma_1 + \Xi_1) + 2(\Delta_1 + 2\Sigma_1^* + \Xi_1^*)$		1	ϵ'		$0.08 \pm 0.13\%$
$c_{405}^{8,1}$	$(-N_1 + 10\Sigma_1 + 11\Xi_1 + 8\sqrt{3}\,\Lambda\Sigma^0) - 2(\Delta_1 + 2\Sigma_1^* + \Xi_1^*)$		1	ϵ'		—
$c_{405}^{27,1}$	$(N_1 - \Xi_1 + 2\sqrt{3}\,\Lambda\Sigma^0) + 2(\Delta_1 - 3\Sigma_1^* - 4\Xi_1^*)$		$1/N_c$	$\epsilon'\epsilon$		—
	$25(\Sigma_1 + \Xi_1) - 3(4\Sigma_1^* - 3\Xi_1^*)$		1	ϵ'		$0.36 \pm 0.02\%$
	$N_1 - \Xi_1$		1	ϵ'		$0.23 \pm 0.03\%$
	$5(2N_1 + \Sigma_1 - \Xi_1) - 3(4\Sigma_1^* - 3\Xi_1^*)$		$*$	$*$		$0.005 \pm 0.018\%$
	$\Sigma_1^* - 2\Xi_1^*$		$1/N_c$	$\epsilon'\epsilon$		$0.04 \pm 0.03\%$
		$I=2$				
$c_{(2)}^{27,2}$	$35\Sigma_2 - 2(3\Delta_2 + \Sigma_2^*)$		$1/N_c$	ϵ''	No	$0.12 \pm 0.03\%$
$c_{(3)}^{27,2}$	$7\Sigma_2 - (3\Delta_2 + \Sigma_2^*)$		$1/N_c$	ϵ''	$\checkmark$	$0.16 \pm 0.06\%$
$c_{(3)}^{64,2}$	$\Delta_2 - 2\Sigma_2^*$		$1/N_c^2$	$\epsilon''\epsilon$	$\checkmark$	$0.20 \pm 0.09\%$
$c_{405}^{27,2}$	$\Sigma_2 + 2(3\Delta_2 + \Sigma_2^*)$		$1/N_c$	ϵ''		$0.26 \pm 0.15\%$
		$I=3$				
$c_{(3)}^{64,3}$	Δ_3		$1/N_c^2$	$\epsilon''\epsilon'$	$\checkmark$	$0\%\text{(fixed)}$

FIG. 1. $SU(2F)$ spin-flavor representation for ground-state baryons. The Young tableau has N_c boxes.

FIG. 2. $SU(F)$ flavor representations for the tower of baryon states with $J = \frac{1}{2}$, $\frac{3}{2}$, ..., $\frac{N_c}{2}$. Each Young tableau has N_c boxes.

FIG. 3. Weight diagram for the $SU(3)$ flavor representation of the spin-$\frac{1}{2}$ baryons. The long side of the weight diagram contains $\frac{1}{2}(N_c + 1)$ weights. The numbers denote the multiplicity of the weights.

FIG. 4. Weight diagram for the $SU(3)$ flavor representation of the spin-$\frac{3}{2}$ baryons. The long side of the weight diagram contains $\frac{1}{2}(N_c - 1)$ weights. The numbers denote the multiplicity of the weights.
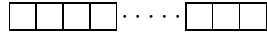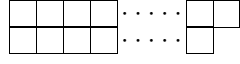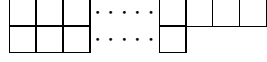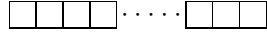
Figure 1

$J = \frac{1}{2}$

$J = \frac{3}{2}$

.
.
.

$J = \frac{N_c}{2}$

Figure 2

```
            1      1
         1      2      1
      1      2      2      1
   1      2      2      2      1
1      2      2      2      2      1
   1   2    2    2    2    2   1
      1   2   2   2   2   2   2   1
   1    2    2    2    2    2    2   1
      1   2   2   2   2   2   2   2   1
1   2   2   2   2   2   2   2   2   1
   1    1    1    1    1    1    1    1    1
```

Figure 3

```
                1    1    1    1

            1    2    2    2    1

        1    2    3    3    2    1

        1    2    3    4    3    2    1

    1    2    3    4    4    3    2    1

  1    2    3    4    4    4    3    2    1

1    2    3    4    4    4    4    3    2    1

  1    2    3    3    3    3    3    2    1

    1    2    2    2    2    2    2    1

        1    1    1    1    1    1    1
```

Figure 4